Math 132: Differential Topology

§ Oriented intersection number

Consider oriented manifolds M, N, P, M compact, $P \overset{\text{closed}}{\subset} N$, $\dim M + \dim P = \dim N$.

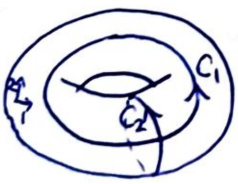
If $f: M \to N$ is transversal to P, then $f^{-1}(P)$ is a finite number of points, each with an orientation number $\in \{\pm 1\}$.

Define the intersection number $I(f, P)$ to be the sum of these orientation numbers. $\in \mathbb{Z}$

The orientation number at $x \in f^{-1}(P)$ is $+1$ (resp. -1) if the isomorphism $df_x(T_x M) \oplus T_{f(x)} P = T_{f(x)} N$ is orientation preserving (resp. reversing).

Note, the ordering is important!

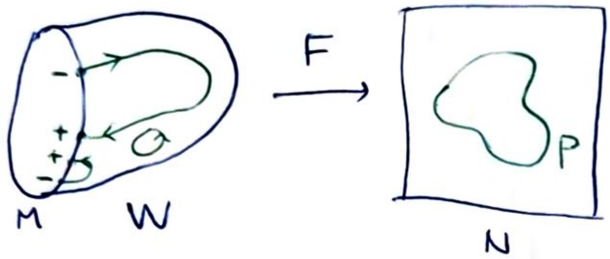
Ex



$\rightsquigarrow$ $I(C_1, C_2) = 1$, $I(C_2, C_1) = -1$

Prop If $M = \partial W$ for some compact oriented W and $f: M \to N$ extends to W, then $I(f, P) = 0$ for any closed $P \subset N$ of complementary dimension.

The proof is the same as the analogous thm in mod 2 intersection theory. Here, $F^{-1}(P)$ is a compact oriented 1-mfld, so the signed count of $\partial F^{-1}(P)$ is 0.



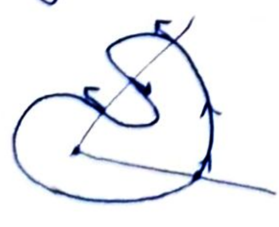
Prop Homotopic maps have the same intersection numbers.

Again, the proof is analogous. If $F: I \times M \to N$ is a homotopy between $f_0, f_1: M \to N$, then $0 = I(\partial F, P) = I(f_1, P) - I(f_2, P)$

$\partial(I \times M) = M_1 - M_0$, so $\partial F^{-1}(P) = f_1^{-1}(P) - f_0^{-1}(P)$.

In the same manner, when N is connected and $\dim N = \dim M$, we define the degree of $f: M \to N$ to be $\deg(f) = I(f, \{y\})$, for any point $y \in N$ (positively oriented).

Ex (Winding number)



Ex $S^1 \to S^1$, $z \mapsto z^m$ has degree $m \in \mathbb{Z}$

$\Rightarrow z^m$ and $z^{m'}$ (as maps $S^1 \to S^1$) are not homotopic to each other whenever $m \neq m'$.

The following reformulation is useful:

Let M_1, M_2 be compact, $\dim M_1 + \dim M_2 = \dim N$.

For any $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$, we say they're transversal,

$f_1 \pitchfork f_2$, if $(df_1)_{x_1}(T_{x_1}M_1) \oplus (df_2)_{x_2}(T_{x_2}M_2) = T_yN$ ⋯ ⊛

whenever $f_1(x_1) = y = f_2(x_2)$.

Basic linear algebra shows $f_1 \pitchfork f_2 \iff f_1 \times f_2 \pitchfork \Delta$

where $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ and $\Delta \subset N \times N$ is the diagonal.

Define $I(f_1, f_2)$ to be the sum of local intersection numbers, each of which is given by $+1$ (resp. -1) if the isomorphism ⊛ is orientation preserving (resp. reversing).

Prop $I(f_1, f_2) = (-1)^{\dim(M_2)} I(f_1 \times f_2, \Delta)$.

proof) This is linear algebra again.

Let $U = (df_1)_{x_1}(T_{x_1}M_1)$ and $V = (df_2)_{x_2}(T_{x_2}M_2)$, and let $\{u_1, \cdots, u_k\}$ and $\{v_1, \cdots, v_\ell\}$ be their positively oriented ordered bases.

We may assume $\{u_1, \cdots, u_k, v_1, \cdots, v_\ell\}$ is positively oriented for T_yN.

Then, $\operatorname{sign}\{\underbrace{(u_1,0), \cdots, (u_k,0), (0,v_1), \cdots, (0,v_\ell)}_{U \times V}, \underbrace{(u_1,u_1), \cdots, (u_k,u_k), (v_1,v_1), \cdots, (v_\ell,v_\ell)}_{\Delta}\}$

$= \operatorname{sign}\{(u_1,0), \cdots, (u_k,0), (0,v_1), \cdots, (0,v_\ell), (0,u_1), \cdots, (0,u_k), (v_1,0), \cdots, (v_\ell,0)\}$

$= (-1)^{\ell(\ell+k)+\ell k} \operatorname{sign}\{(u_1,0), \cdots, (u_k,0), (v_1,0), \cdots, (v_\ell,0), (0,u_1), \cdots, (0,u_k), (0,v_1), \cdots, (0,v_\ell)\}$

$= (-1)^\ell$. ∎

<u>Prop</u> If f_0 and g_0 are homotopic to f_1 and g_1, respectively,
then $I(f_0, g_0) = I(f_1, g_1)$

proof) $f_t \times g_t$ is a homotopy between $f_0 \times g_0$ and $f_1 \times g_1$. ■

In particular, if $M_1, M_2 \subset N$ are compact submanifolds of complementary dimension, then $I(M_1, M_2)$ is invariant under homotopy of either M_1 or M_2.

Note, from the definition, $I(M_1, M_2) = (-1)^{(\dim M_1)(\dim M_2)} I(M_2, M_1)$.

<u>Def</u> If M is a compact, ~~oriented~~ (orientable) manifold, its <u>Euler characteristic</u> $\chi(M)$ is defined to be $I(\Delta, \Delta)$, where $\Delta \subset M \times M$ is the diagonal.

We immediately have:

<u>Prop</u> The self-intersection number of any <u>odd-dimensional</u> compact oriented mfld is 0. In particular, <u>$\chi(M) = 0$ for any odd-dim'l cpt oriented mfld.</u>

Recall that $I_{\mathbb{Z}/2}(C, C) = 1$ for the central circle C in the Möbius strip.
This shows that <u>Möbius strip is not orientable</u> (for if it were, then
$I_{\mathbb{Z}/2}(C, C) = I(C, C) \bmod 2 = 0$).